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Off-shell representations of supersymmetry with central charges

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Abstract. We determine the super-spin and internal symmetry Casimirs for the four-dimensional global extended supersymmetry algebra with central charges, for both the non-degenerate and degenerate case ('spin-reducing'), and show how these two cases are related. The nature of the off-shell symmetries and representations on superfields are analysed. The 'super-tableau calculus' is extended to the central charge case, and the constraints on superfields required to produce the fundamental irrep and others are determined (for even N); prepotentials are discovered for $N = 2$.

1. Introduction

One of the outstanding problems in the construction of a unified theory of the forces of nature including gravity is that of cancelling the ultraviolet divergence arising in quantum calculations. Supersymmetry appears to give a hope of incorporating such cancellation, though this has only yet been shown explicitly at the one- and two-loop level (Duff 1982). In order to use most efficiently the Bose–Fermi symmetry producing such cancellations, superfield techniques have been developed for the case of un-extended supersymmetry (Grisaru 1982). In order to preserve supersymmetry explicitly in the extended cases and also achieve maximal use of the cancellation mechanism, it is necessary to have a suitably extended superspace version of the extended supersymmetric theory. Thus for N-extended supergravity (N-SGR) it appears most efficient for ultraviolet divergence cancellation to have a maximally extended superspace formulation. However, in spite of many efforts to achieve such a theory this has so far proved impossible for $N \geqslant 3$.

The existence of the '$N = 3$ barrier' has been explained recently (Rivelles and Taylor 1981, Taylor 1982a) in terms of counting of fermion modes available in the irreducible representations (irreps) of N-extended supersymmetry (N-SUSY). In order that the correct physical spectrum for N-SGR be achieved from a set of such irreps, certain field redefinitions (Rivelles and Taylor 1982a) are necessary among the field components of the irreps. The $N \geqslant 3$ no-go theorems of Rivelles and Taylor (1981) and Taylor (1982a) were based on the impossibility of using such field redefinitions so that all fermions other than the desired physical fields could be combined into auxiliary spinors and so have vanishing value on-shell by virtue of their field equations. This impossibility was shown explicitly (Rivelles and Taylor 1981, Taylor 1982a) for $N = 3$ and 4, though more recently the proof has been considerably simplified and extended to higher dimensions (Rivelles and Taylor 1983a), with the result that no explicitly supersymmetric off-shell version of N-SGR can be constructed for $N \geqslant 3$.

The proof of the $N \geqslant 3$ no-go theorem depends on certain assumptions, the most crucial of which is that no central charges are present in the N-SUSY algebra $\mathcal{S}_N$. Such charges commute with all operators of $\mathcal{S}_N$ except the internal symmetry generators, and can modify the representation content of $\mathcal{S}_N$ considerably in the degenerate or 'spin-reducing' case when the irreps of $\mathcal{S}_{N/2}$ become relevant (for N even). The use of such irreps can circumvent the $N \geqslant 3$ no-go theorem, and one of us produced (Taylor 1981a, b) candidate irreps for a satisfactory off-shell formulation of N-SGR for $N = 3$, 4, 5, 6 and 8. These candidates were somewhat schematic and required an analysis of the SUSY transformation laws of the redefined fields in order to show that there are no non-local terms arising (which would prevent non-linearisation). Such an analysis has now been given in detail for $N = 1$ (Rivelles and Taylor 1982b, 1983b) and 2 (Rivelles and Taylor 1983c), with new auxiliary field sets being discovered by this approach. In particular, certain of the $N = 2$ solutions involved central charges, so are interesting analogues of the situation we must meet for $N \geqslant 3$. In order to extend this programme to such higher values of N, the component methods used for $N = 1$ and 2 become rather cumbersome due to increasing numbers of component fields in irreps of $\mathcal{S}_N$: the 'component explosion'. Although we still hope to analyse $N = 4$ SGR by such techniques, the case of $N = 8$ (and $N = 4$ super-Yang–Mills; $N = 4$ SYM) appears very difficult, and we will have to adapt superfield techniques to the problem.

Much is known about the representation of N-SUSY on superfields in the absence of central charges (Bufton and Taylor 1983a, Ferrara *et al* 1981, Nahm 1978, Pickup and Taylor 1981, Rittenberg and Sokatchev 1981, Salam and Strathdee 1974). We wish to develop a similar understanding in the presence of central charges. There is a growing but rather scattered literature on the subject (Fayet 1975, 1979, Sohnius 1978, Sohnius *et al* 1981, Taylor 1980), especially for the case of $N = 2$. We wish to give here a complete account of the representation theory of central charge SUSYs on superfields which will allow application to the programme of constructing N-SGRs for $N \geqslant 3$.

One approach to central charges is to regard them as components of momenta in extra dimensions. This possibility leads to the use of a higher-dimensional framework in which to build N-SGRs *ab initio*. This is particularly appealing since then only lower N values need to be discussed; in $d = 11$ only $N = 1$ is possible. In spite of the fact that much interest is presently being engendered by the resuscitation of Kaluza–Klein theory (Salam and Strathdee 1981), and especially the addition of spinors (Witten 1981) and the possible and ultraviolet divergence cancellations in these higher dimensions (Duff and Toms 1982), we will not follow the higher-dimensional trail at this point. There are various reasons for this.

The first point is that we are specifically interested in the quantisation of N-SGR in four dimensions. The quantum properties of higher-dimensional theories are of interest, but we are here trying to exploit the expected cancellations due to N-SUSY in four dimensions alone. Thus certain features of the classical theory, such as the hidden symmetries, may be usefully analysed by means of higher dimensions. However, the quantisation of the four-dimensional theory requires its explicit construction as a classical theory in that number of dimensions before quantisation is to be achieved.

Another reason for restricting ourselves to four dimensions is that the ultraviolet divergence cancellation known to occur on-shell for $N = 4$ SYM in $d = 4$ up to three loops (Avdeev *et al* 1980, Caswell and Zanon 1980, Grisaru *et al* 1980) does not seem to do so even at the one-loop level for $N = 1$ SYM in $d = 10$ (Ragiadakos and Taylor 1982). This is a hint that the ultraviolet divergences of quantised higher-dimensional super-theories may have the virulence expected of them and that the four-dimensional cancellations do not persist in higher dimensions.

A third and related point is that the construction of off-shell N-SGRs in higher dimensions also has difficulties. The higher-dimensional no-go theorems (Rivelles and Taylor 1983a) show that there is only one case in which no central charges are needed, and that is $N = 1$ SGR in $d = 10$. This corresponds to $N = 4$ SGR plus six $N = 4$ SYMs in $d = 4$; a linearised superfield version of this possibility has already been presented (Howe *et al* 1982). However, $N = 1$ SGR in $d = 11$ requires central charges in its algebra in order to obtain an off-shell theory. This clearly complicates the situation, and we can regard our analysis of central charges in $d = 4$ as a preliminary to that for $d = 11$.

A further feature in favour of working directly in $d = 4$ is that the central charge structure can be made explicit. If we consider this structure in the higher-dimensional framework, it is necessary to analyse how a suitable dimensional reduction can be achieved to produce the required structure in $d = 4$. At least two central charges appear necessary for $N = 4$ SGR (Bufton and Taylor 1983b), and no known dimensional reduction can achieve this in a general manner (Sohnius *et al* 1981). We can regard our analysis as giving hints as to how higher dimensions may be entering by starting from the $d = 4$ theory with suitable central charges and interpret the corresponding theory as coming from one in a higher dimension:

Some of the ideas used here have already appeared elsewhere (Taylor 1982b), but we will present a self-contained account which has a different emphasis. In particular, we will try to develop properties parallel to those of the non-central charge case. We start our programme by constructing the Casimirs of the N-SUSY algebra with central charges. This is done for the Poincaré and internal symmetry algebras in § 2. We find that there are two cases to consider, already known in the literature (Sohnius 1978, Sohnius *et al* 1981, Taylor 1980), which we call the non-degenerate and degenerate cases. The former is discussed in detail in § 3 and the latter in § 4. We develop the theory of representations on superfields (SFs), and in particular extend the super-tableau calculus so as to give the constraints for various irreps in suitable SFs. We solve these in § 5 in terms of unconstrained prepotentials. We discuss future problems in the final section.

2. Casimirs for central charge SUSY (non-degenerate)

The N-SUSY algebra $\mathcal{S}_N$ will be chosen to have chiral SUSY generators $S_{\alpha+}{}^{l}$ and complex conjugates $S_{\alpha-l}$, with $1 \leq l \leq N$. The other generators are $J_{\mu\nu}$ and P_μ constituting the Poincaré algebra and internal symmetry generators which we will introduce shortly. The anticommutators of the chiral generators are

$$[S_{\alpha+}{}^{l}, S_{\beta-m}]_+ = -2(\not{P}C)_{\alpha+\beta-}\delta^{l}{}_{m} \qquad [S_{\alpha+}{}^{l}, S_{\beta+}{}^{m}]_+ = 2C_{\alpha+\beta+}Z^{lm} \qquad (1a, b)$$

where C is the charge conjugation matrix and Z^{lm} are a set of $\frac{1}{2}N(N-1)$ complex generators commuting with $S_{\alpha+}{}^{l}$, $S_{\alpha-l}$, P_μ and $J_{\mu\nu}$. We note also the conjugate of $(1b)$:

$$[S_{\alpha-l}, S_{\beta-m}]_+ = 2C_{\alpha-\beta-}Z^{*lm} \qquad (1c)$$

and also use

$$[J_{\mu\nu}, S_{\alpha+}{}^{l}]_{-} = \mathrm{i}(\sigma_{\mu\nu}S_{+}{}^{l})_{\alpha+} \tag{1d}$$

where we take the metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]_-$. We wish to modify the Pauli–Lubanski vector $W_\mu = \frac{1}{2}\Sigma_{\mu\nu\lambda\sigma}J^{\nu\lambda}P^\sigma$ in a similar manner to that when $Z^{ij} = 0$ by addition of further terms to produce a conserved vector C which commutes with $S_{\alpha+}{}^{l}$. The available real vectors are $K_\mu = (\bar{S}^l_+\gamma_{\mu'}S_{-l} - \bar{S}_{-l}\gamma_{\mu'}S_+{}^l)$ and $J_\mu = p^\nu(\bar{S}_+{}^l\sigma_{\mu\nu}S^m_+Z^{*lm} - \bar{S}_{-l}\sigma_{\mu\nu}S_{-m}Z^{lm})$ (where $\gamma_{\mu'} = \gamma_\mu - p_\mu \not{p}/p^2$, so $p^\mu\gamma_{\mu'} = 0$), so we consider the general linear combination $C_\mu = W_\mu - \alpha K_\mu - (\beta/p^2)J_\mu$. From (1) we have, after some algebra, that

$$[C_\mu, S_{\alpha+}{}^{i}]_- = \tfrac{1}{2}(\gamma_{\mu'}\not{p}S_+{}^l)_{\alpha+}[(1-8\alpha)\delta^i{}_l + (4\beta/p^2)Z^{*lm}Z^{mi}] + (\gamma_{\mu'}S_{-l})_{\alpha+}Z^{il}(4\alpha - 2\beta). \tag{2}$$

In order for (2) to vanish we require

$$\beta = 2\alpha \qquad Z^{*lm}Z^{mi} = \kappa p^2\delta^i{}_l \tag{3a, b}$$

where $\kappa = (1 - 1/8\alpha)$. Whilst (3*a*) is trivial, (3*b*) is decidedly not so but is extremely restrictive for $N > 2$. We turn to the solution of (3*b*) for various $N > 2$ shortly, but note especially that for $N = 2$ we always have

$$Z^{ij} = \varepsilon^{ij}Z \tag{4a}$$

with $\varepsilon^{12} = +1$, $\varepsilon_{12} = -1$, and (3*b*) then becomes

$$-|Z|^2 = \kappa p^2 \tag{5}$$

which would be a massless condition in $d = 6$ if $\kappa = 1$.

For general N we may use the number κ determined by (3*b*) to give

$$C_\mu = W_\mu + [8(\kappa - 1)]^{-1}\kappa_\mu + [4p^2(\kappa - 1)]^{-1}J_\mu \tag{6}$$

where $(\kappa - 1)$ in (6) takes the value $(N^{-1}Z^{*lm}Z^{ml} - p^2)(p^2)^{-1}$. We note that this solution only exists for $\kappa \neq 1$; which we will assume in this section. We term this the non-degenerate case, whilst we will consider the degenerate case of $\kappa = 1$ in § 3. For the former case, $C^2 = C_\mu C^\mu$ will be a Casimir of the algebra $(P_\mu, J_{\mu\nu}, S_{\alpha+}{}^l, S_{\alpha-l})$. Furthermore, in the rest frame $P_\mu = (M, 0)$ we can show, after further algebra, that C_i satisfies the SU(2) algebra. Thus C^2 will take the values $p^2Y(Y+1)$ for $Y = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$. For $Z^{ij} = 0$ the value of C_μ in (6) reduces to that in Taylor 1982b, with $J_\mu = 0 = \kappa$. We have thus a suitable generalisation of superspin to the central charge situation (modulo internal symmetries) with an identical superspin spectrum to the non-central charge case.

A conceptual problem arises when we turn to define the internal symmetry of the algebra (1). This is because the Z^{ij} are operators without fixed values. Thus the Z^{ij} can define different directions in GL(N, C) corresponding to their value on different irreps. This situation has been analysed elsewhere (Ferrara and Savoy 1982) by means of a mapping to a canonical form of Z^{ij}, but we cannot perform that here due to the operator character of Z_{ij}; such a canonical approach is only valid for on-shell representations. However, we may use (3*b*) to define a pseudo-symplectic metric Ω^{ij} by

$$\Omega^{ij} = Z^{ij}/\sqrt{\kappa p^2} \tag{7a}$$

where we only consider $\kappa \neq 0$. Then (3*b*) may be rewritten as

$$\Omega^{*ij}\Omega^{jk} = \delta_i{}^k \tag{7b}$$

and so we may take Ω^{*ij} as the lowered symplectic metric Ω_{ij}. Since Ω^{ij} is an operator we do not have a unique symplectic symmetry, but only the intersection of the symplectic groups USp(N) for the various numerical matrices that Ω^{ij} may become on given representations. This intersection may even be null, as in the case of five or more central charges (in different irreps) for $N=4$, or six or more central charges for $N=8$.

We may now define the pseudo-symplectic generators $A_r^{\ s}$ with

$$(A_r^{\ s})^\dagger = A_s^{\ r} \tag{8a}$$

$$[A_r^{\ s}, A_t^{\ u}]_- = \mathrm{i}(\delta_r^{\ u} A_t^{\ s} - \delta_t^{\ s} A_r^{\ u} + \Omega^{su}\Omega_{rl}A_t^{\ l} - \Omega_{rt}\Omega^{sl}A_l^{\ u}) \tag{8b}$$

$$[A_r^{\ s}, S_{\alpha+}^{\ \ l}]_- = \mathrm{i}(\delta_r^{\ l} S_{\alpha+}^{\ \ s} - \Omega^{sl}\Omega_{mr} S_{\alpha+}^{\ \ m}). \tag{8c}$$

We note that C_μ defined in (6) is invariant under this pseudo-symplectic group.

We now wish to extend $A_r^{\ s}$ to operators $T_r^{\ s}$ in a similar manner to that of W_μ to C_μ so that $T_r^{\ s}$ commutes with $S_{\alpha+}^{\ \ l}$ and $S_{\alpha-l}$; the invariant trace of powers of $T_r^{\ s}$ will then give the pseudo-symplectic group generators in an identical fashion to that for $\mathcal{S}_N$ without central charges. The appropriate pseudo-symplectic Hermitian tensors we may add to $A_r^{\ s}$ are

$$U_r^{\ s} = \mathrm{i}(\bar{S}_+^{\ s}\not{p}S_{-r} + 2p^2\delta_r^{\ 2}) \tag{9a}$$

$$V_r^{\ s} = \mathrm{i}(\bar{S}_+^{\ s}S_+^{\ m}Z^{*mr} - \bar{S}_{-r}S_{-m}Z^{ms}) \tag{9b}$$

$$W_r^{\ s} = \mathrm{i}Z^{*rn}(\bar{S}_+^{\ n}\not{p}S_{-m} + 2p^2\delta_m^{\ n})Z^{ms}. \tag{9c}$$

Then we find that the unique linear combination of $A_r^{\ s}$, $U_r^{\ s}$, $V_r^{\ s}$ and $W_r^{\ s}$ which commutes with $S_{\alpha+}^{\ \ l}$ is

$$T_r^{\ s} = A_r^{\ s} + [2p^2(\kappa-1)]^{-1}[-U_r^{\ s} + V_r^{\ s} + (\kappa p^2)^{-1}W_r^{\ s}]. \tag{10}$$

We again see that $T_r^{\ s}$ is only properly defined for $\kappa \neq 1$, though we will prove shortly that its value as $\kappa \to 1$ allows us to discuss the degenerate case most satisfactorily. We may also calculate $[T_r^{\ s}, T_t^{\ u}]$ and find, after some algebra, that is has the value (8b). Thus the powers

$$T_r = (T_{r_1}^{\ \ r_2} T_{r_2}^{\ \ r_3} \ldots T_{r_n}^{\ \ r_1}) \tag{11}$$

form the usual Casimirs of the pseudo-symplectic symmetry. We thus see that all the usual Casimirs of the non-central charge SUSY algebra may be extended to the non-degenerate case. We have to remember, however, that the final symplectic symmetry is obtained by a suitable intersection of the symplectic symmetries of different representations.

Let us consider this in a little detail for $N=4$. In that case we may take the six real generators of SU(4) as the real 4×4 antisymmetric matrices $\boldsymbol{\alpha}^{ij}$, $\boldsymbol{\beta}^{ij}$ with $[\alpha_l, \beta_m]_- = 0$, $\alpha_l\alpha_m = -\varepsilon_{lmn}\alpha_n$, $\beta_l\beta_m = -\varepsilon_{lmn}\beta_n$, $\beta^2{}_l = \alpha^2{}_l = -1$ $(1 \leq l, m, n \leq 3)$. We may write

$$Z^{ij} = \boldsymbol{Z}\cdot\boldsymbol{\alpha}^{ij} + \boldsymbol{Z}'\cdot\boldsymbol{\beta}^{ij}. \tag{12}$$

We may take representations with an increasing number of the set $(\boldsymbol{Z}, \boldsymbol{Z}')$ non-zero. The symplectic symmetry has symmetric generators anticommuting with Z^{ij} and antisymmetric generators commuting with Z^{ij}. This is because if M is an infinitesimal matrix the tranformation $S_{\alpha+} \to (1+M)S_{\alpha+}$ leaves (1b) unchanged provided $(1+M)Z(1+M)^{\mathrm{T}} = Z$, or $MZ = -ZM^{\mathrm{T}}$. Thus if $M = M^{\mathrm{T}}$ then $[M, Z]_+ = 0$ and if $M =$

$-M^{\mathrm{T}}$, $[M, Z]_- = 0$. Along the direction Z_1 the ten USp(4) generators are

$$(\alpha_1, \boldsymbol{\beta}; \alpha_2\boldsymbol{\beta}, \alpha_3\boldsymbol{\beta}) \tag{13a}$$

where we have divided the set into antisymmetric and symmetric ones respectively. Similarly along Z_2, Z_3, Z'_1, Z'_2, Z'_3 the generators are respectively

$$(\alpha_2, \boldsymbol{\beta}; \alpha_1\boldsymbol{\beta}, \alpha_3\boldsymbol{\beta}); \quad (\alpha_3, \boldsymbol{\beta}; \alpha_1\boldsymbol{\beta}, \alpha_2\boldsymbol{\beta}); \quad (\beta_1, \boldsymbol{\alpha}; \beta_2\boldsymbol{\alpha}, \beta_3\boldsymbol{\alpha});$$
$$(\beta_2, \boldsymbol{\alpha}; \beta_1\boldsymbol{\alpha}, \beta_3\boldsymbol{\alpha}); \quad (\beta_3, \boldsymbol{\alpha}; \beta_1\boldsymbol{\alpha}, \beta_2\boldsymbol{\alpha}). \tag{13b}$$

If two different irreps are present with different directions for Z^{ij} then the total symmetry will be generated by the intersection of the sets in (13*a*) and (13*b*). Thus if there are central charges Z_1 and Z_2 the common generators are $(\boldsymbol{\beta}; \alpha_3\beta)$ with group USp(2)×USp(2); if there are three central charges the common generators are $(\boldsymbol{\beta})$, generating SU(2). The chain of symmetries is, for increasing numbers of central charges,

$$\mathrm{SU}(4) \supset \mathrm{USp}(4) \supset \mathrm{USp}(2) \times \mathrm{USp}(2) \supset \mathrm{USp}(2) \supset \mathrm{U}(1) \supset \varnothing \tag{14}$$

where the first group in (14) is the symmetry with no central charges and the last with five or six. A similar situation occurs for $N = 8$ and higher N.

Before we consider such values of N let us analyse the condition (3*b*) for general N. We have already remarked that for $N = 2$ (3*b*) reduces to (5). For $N = 4$ we may use (12) so that the LHS of (3*b*) becomes

$$Z_l^* Z_m (\alpha_l \alpha_m)^{ij} + Z_l'^* Z'_m (\beta_l \beta_m)^{ij} + (Z_l^* Z'_m + Z_l Z_m'^*)(\alpha_2 \beta_m)^{ij}.$$

Then (3*b*) is satisfied if either

$$Z_l^* = Z_l, Z_l'^* = -Z'_l \qquad \text{or} \qquad Z_l^* = -Z_l, \quad Z_l'^* = Z'_l \tag{12a}$$

and in either case we need

$$\sum_{l=1}^{3} (Z_l^* Z_l + Z_l'^* Z'_l) = -\kappa p^2. \tag{12b}$$

In either of the cases (12*a*) there is a maximum of six real central charges and if normalised by $\kappa^{-1/2}$ and with inclusion of the four original space–time variables they satisfy the massless wave equation in up to ten dimensions. That $N = 4$ supersymmetry can arise from simple supersymmetry in $d = 10$ is well known. What we have shown here is that in order for there to be a suitable superspin operator in four dimensions, with the properties it possesses without central charges, there can be at most six such extra dimensions.

A similar situation arises for $N = 8$. For that we can introduce a set of matrices Γ_l similar to the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of $N = 4$, as the generators of the seven-dimensional Clifford algebra with

$$[\Gamma_l, \Gamma_m]_+ = -2\delta_{lm} \tag{13c}$$

with which we include the 21 generators $[\Gamma_l, \Gamma_m]_- = \Gamma_{lm}$ of SO(7). The general form of the central charge matrix is thus

$$Z = \sum_{l=1}^{7} Z_l \Gamma_l + \sum_{l<m} Z_{lm} \Gamma_{lm}. \tag{13d}$$

The condition (3*b*) requires that

$$Z_{lm}=0 \qquad Z_l^*=\pm Z_l \tag{13e}$$

giving a maximum number of seven central charges; this corresponds to the well known seven extra dimensions for $N=8$ SGR. The chain corresponding to (14) for $N=8$ is

$$\mathrm{SU}(8)\supset \mathrm{USp}(8)\supset \mathrm{USp}(4)\times \mathrm{USp}(4)\supset \mathrm{SU}(2)\supset \mathrm{U}(1)\supset \varnothing . \tag{14a}$$

The solution of (3*b*) for general N is not known, though it appears to be related to the problem of sums of squares (Taussky 1970).

3. Casimirs in the degenerate case

We now turn to the case $\kappa=1$ for which the values of C_μ of (6) and T_r^s of (10) appear to become singular. The way to deal with this singularity can be seen if we rewrite T_r^s as

$$T_r^s=A_r^s+[\mathrm{i}/2p^2(\kappa-1)][\bar S_+{}^s(S_+{}^mZ^{*mr}-\not p S_{-r})+(\kappa p^2)^{-1}Z^{*rn}Z^{ms}(\bar S_+{}^n\not p-Z^{nl}\bar S_{-l})S_{-m}]. \tag{15}$$

For T_r^s to have a finite limit as $\kappa\to 1$ we must require the vanishing of the terms in parentheses in (15), and so reach the condition

$$\not p S_{-r}-S_+{}^mZ^{*mr}=0. \tag{16}$$

Condition (16) takes the form of a Dirac equation with 'mass' $\sim Z^*$. When (16) and its complex conjugate are satisfied then $\kappa=1$ and C_μ of (6) and T_r^s of (10) reduce to

$$C_\mu=W_\mu+(1/8p^2)\bar S_+{}^l\gamma_\mu\not p S_+{}^mZ^{*lm} \tag{17a}$$

$$T_r^s=A_r^s+(\mathrm{i}/2p^2)\bar S_+{}^sS_+{}^mZ^{*mr}. \tag{17b}$$

In the rest frame $\boldsymbol{C}$ still satisfies an SU(2) algebra and T_r^s generates USp(N).

The condition $\kappa=1$ and the related Dirac condition (16) now become constraints on the algebra which require some discussion. For $\kappa\neq 1$ the expressions (6) and (10) are valid without further restriction. They thus parametrise representations of the SUSY algebra (1) in the same way as for the non-central charge case. However, the degenerate case has fewer elements when (16) is used. Since $\kappa=1$ arises from (16) this latter is the sole independent constraint. It states that we can replace S_{-r} by $\not p^{-1}S_+{}^mZ^{*mr}$, thus leaving only the independent elements $S_+{}^l$. Moreover, it is straightforward to prove that (1*a*) arises from (1*b*) and (16). Thus the Fermi part of the SUSY algebra has been reduced to half its original size by the imposition of the constraint (16).

The use of the Dirac constraint in this way allows us to extend the analysis of the spin content of on-shell irreps to the off-shell situation. This is important for the construction of field theories incorporating these irreps, such as the cases of $N=2$ and 4 SYM and $N\geqslant 2$ SGR. The most essential feature of 'spin reduction'—the reduction of the maximum Poincaré spin in any irrep to half of that for $\mathscr{S}_N$ without central charges (for even N)—occurs in the degenerate central charge case. That is because the reduction of the number of Fermi generators decreases by one half the number of available fermion creation operators. On-mass-shell analysis (Ferrara and Savoy 1982) shows that this does indeed occur. The general analysis of irreps using basis functions (Bufton and Taylor 1982a) is thus expected to give only half the maximal spin in any irrep compared with the non-degenerate case. We will see this in more detail shortly when we consider superfield representations.

Before we turn to this we must reconsider the Dirac condition (16) further. For central charges satisfying (3b) we can write (as discussed at the end of § 2)

$$Z^{ij} = \sum_{l=1}^{D} (\Gamma_l)^{ij} X_l \tag{18}$$

where Γ_l belong to a suitable Clifford algebra, with real X_l, and (3b) becomes, for $\kappa = 1$,

$$\sum_{l=1}^{D} X_l^2 = -p^2. \tag{19}$$

This is the massless wave equation in $(4+D)$ dimensions in the representation $X_l = \partial/\partial x^l$. Masslessness is known to reduce the dimensions of supersymmetry irreps, and so the massless condition (19) may be regarded as the underlying reason for spin reduction. As we have already remarked, spin reduction is the only presently known way of achieving off-mass-shell models of $N \geqslant 3$ SGRs and $N = 4$ SYM, so that masslessness in higher dimensions is clearly of importance for such theories.

4. Superfields in the degenerate case

Superfields provide a highly compact method to describe representations of N-SUSY, especially for higher N. However, they are reducible and since they contain irreps unwanted for the construction of N-SGRs and N-SYMs it is necessary to impose constraints on the superfields (SFs) to exclude the unwanted irreps. The ultimate goal is to construct an unconstrained SF form of the above theories, where the unwanted modes correspond to gauge degrees of freedom; such an aim seems achievable only through constrained SFs as an intermediate step.

To determine constraints on SFs which give solely irreps, let us start by introducing our superspace with Bose variables x^μ, z_{ij}, $z_{im}{}^*$ and Fermi variables $\theta_{\alpha+l}$, $\theta_{\alpha-}{}^l$. Then we may represent $S_{\alpha+}{}^l$, $S_{\alpha-l}$ satisfying (1) as

$$S_{\alpha+}{}^l = \mathrm{i}(\partial/\partial\bar\theta^{\alpha+}{}_l + \mathrm{i}(\not\partial\theta_-{}^l)_{\alpha+} + \theta_{\alpha+m}\,\partial/\partial z_{lm}) \tag{20a}$$

$$S_{\alpha-l} = -\mathrm{i}(\partial/\partial\bar\theta^{\alpha-l} + \mathrm{i}(\not\partial\theta_{+l})_{\alpha-} + \theta_{\alpha-}{}^m\,\partial/\partial z^*_{lm}) \tag{20b}$$

$$P_\mu = \mathrm{i}\,\partial/\partial x^\mu \qquad J_{\mu\nu} = x_{[\mu}P_{\nu]} - (\bar\theta_{+l}\sigma_{\mu\nu}\,\partial/\partial\bar\theta_{+l} + \bar\theta_-{}^l\sigma_{\mu\nu}\,\partial/\partial\bar\theta_-{}^l) \tag{20c}$$

$$T_r{}^s = \mathrm{i}(\bar\theta_{+r}\,\partial/\partial\bar\theta_{+s} - \bar\theta_-{}^s\,\partial/\partial\bar\theta_-{}^r) - \mathrm{i}(\Omega_{mr}\Omega^{sl}\bar\theta_{+l}\,\partial/\partial\bar\theta_{+m} - \Omega^{ms}\Omega_{rl}\bar\theta_-{}^l\,\partial/\partial\bar\theta_-{}^m) \tag{20d}$$

The Dirac term on the LHS of (16) can then be evaluated to be

$$(\not p S_{-l})_{\alpha+} - S_{\alpha+}{}^m Z^{*ml} = -\mathrm{i}\not p_{\alpha+}{}^{\beta-}(\partial/\partial\bar\theta^{\beta-l} + (\not p^{-1})_{\beta-}{}^{\gamma+} Z^{*ml}\,\partial/\partial\bar\theta^{\gamma+}{}_m) \tag{21}$$

Thus every superfield $\Phi(x^\mu, z_{ij}, z^*{}_{ij}, \theta_{+l}, \theta_-{}^m)$ has to satisfy the condition (16), so can be written as

$$\Phi(x^\mu, z_{ij}, z_{ij}{}^*, \theta_{+l} - Z^{*lm}\not p^{-1}\theta_-{}^m). \tag{22}$$

We must interpret (22) carefully, however, since the differential operators ∂_μ and $\partial/\partial z^*{}_{lm}$ are contained in the SF Φ. We define (22) by the expansion of Φ in powers of the Fermi variable

$$\psi_{+l} = \theta_{+l} - Z^{*lm} \not{p}^{-1} \theta_{-}{}^{m}. \tag{23}$$

Thus we have the definition

$$\Phi(x, z, z^*, \psi_+) = \sum \psi_+{}^n \phi_n(x, z, z^*) \tag{24}$$

where $\psi_+{}^n$ is a symbolic representation of n powers of ψ_+ and the coefficient functions ϕ_n involve appropriate Fermi variables to saturate those on $\psi_+{}^n$. We see immediately that, due to the chiral nature of ψ_+, there are only half as many spinor labels on ϕ_n as if ψ_+ had both chiralities. In other words, the Dirac condition (16) has reduced the number of spins available in any (SF) representation by one half, so spin reduction has occurred explicitly.

It is now straightforward to extend the 'super-tableau calculus' (Howe *et al* 1981a, b) to this degenerate central charge case. The resulting formalism is simpler than for $Z^{ij} = 0$ due to the Dirac condition. This means that we may dispense, for example, with $D_{\alpha - l}$ in preference to $D_{\alpha +}{}^{l}$, and so with boxes with crosses in them; only boxes with dots in them are needed to describe component fields of constrained superfields.

In the case of $N = 2$ the fundamental multiplet with $Y = 0$ is not contained in a scalar superfield, since from the theory of irreps of $N = 1$ SUSY (to which the $N = 2$ case reduces in the spin-reducing case) the $Y = 0$ irrep will be an SU(2) doublet in such a superfield. The $Y = 0$ irrep must thus be in a doublet superfield Φ^i, represented by a single box $\square$, as is known (Sohnius 1978). The constraint which singles out the $Y = 0$ irrep from the $Y = \frac{1}{2}$ irrep in Φ^i can be seen to be

$$\begin{array}{|c|c|}\hline \ & \cdot \\ \hline \end{array} = 0. \tag{25}$$

That this is so follows immediately from the derivation from (25) of the component fields in Φ^i as

$$A^i = \square \qquad \psi = \begin{array}{|c|}\hline \ \\ \hline \cdot \\ \hline \end{array} \qquad F^i = Z\square \qquad \lambda = Z\begin{array}{|c|}\hline \ \\ \hline \cdot \\ \hline \end{array} \tag{26}$$

(where we mean by the tableaux on the RHS of (26) that the SF given there is evaluated at $\theta = z = 0$, and we consider only one Z, for simplicity). Since (26) agrees with the known component fields of the $Y = 0$ irrep, this justifies the constraint (25).

A similar result follows for $N = 4$, where the constrained SF must be an USp(4) 5-plet Φ^{ij}, represented by $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$. The corresponding constraint singling out the $Y = 0$ fundamental irrep is

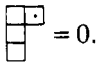
$$\begin{array}{|c|c|}\hline \ & \cdot \\ \hline \ \\ \cline{1-1} \end{array} = 0 \tag{27}$$

and the components will be

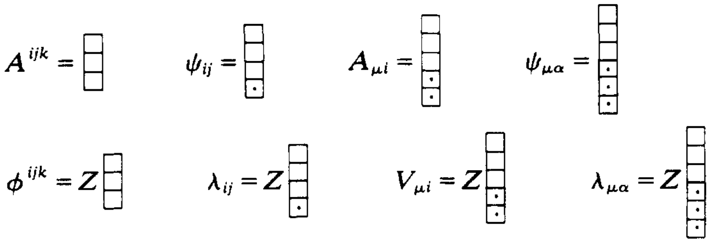
$$A^{ij} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \qquad \psi_i = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \cdot \\ \hline \end{array} \qquad V_\mu = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \cdot \\ \hline \cdot \\ \hline \end{array} \qquad \phi^{ij} = Z\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \qquad \lambda_i = Z\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \cdot \\ \hline \end{array} \qquad A_\mu = Z\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \cdot \\ \hline \cdot \\ \hline \end{array}. \tag{28}$$

Since the highest component field has integer spin it is possible to apply a reality condition so that $\lambda_i = \not{p}\psi_i$ and A^{ij}, ϕ^{ij}, A_μ and V_μ are real.

We may extend the above to $N=6$ with a 14-plet SF Φ^{ijk}, with the constraint singling out the $Y=0$ fundamental irrep being


(29)

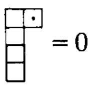
The component content of the $Y=0$ irrep is



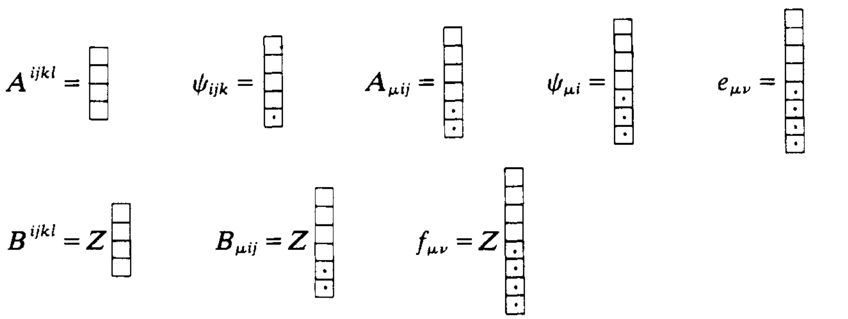
(30)

where A^{ijk} and ϕ^{ijk} are in the 14′-dimensional, ψ_{ij} and λ_{ij} the 14-dimensional, $A_{\mu i}$ and $V_{\mu i}$ the six-dimensional and $\psi_{\mu\alpha}$, $\lambda_{\mu\alpha}$ in the one-dimensional irreps of USp(4) respectively.

For $N=8$ the corresponding SF is the 42-plet SF Φ^{ijkl} with the constraint for the fundamental irrep being

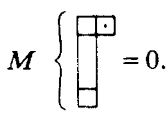

(31)

and the component content



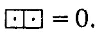
(32)

where a reality condition has already been applied to (32) so that all fermions are Majorana and all bosons are real. We may clearly extend (25), (27), (29) and (31) to arbitrary $N = 2M$, with the SF $\Phi^{i_1 \cdots i_M}$ described by the tableau  $\}$ M and constraint


(33)

The component content will be an obvious generalisation of (26), (28), (30) and (32), with a reality condition being possible for even M.

We may also consider constraints on other superfields to obtain irreps. Thus for $N=2$ we may take the constraint on the scalar SF Φ with


(34)

The component fields will thus be

$$A = \circ \qquad F = Z\circ \qquad \psi_i = \boxdot \qquad A_\mu = \begin{smallmatrix}\boxdot\\ \boxdot\end{smallmatrix} \qquad V_\mu = Z\begin{smallmatrix}\boxdot\\ \boxdot\end{smallmatrix} \tag{35}$$

where we have applied an obvious reality condition. (35) is the $Y = \frac{1}{2}$ gauge supermultiplet with central charge used to construct Abelian $N = 2$ SYM (Sohnius *et al* 1981, Taylor 1980). The other irrep in Φ has $Y = 0$ and is defined by the constraint

$$\begin{smallmatrix}\square\\ \boxdot\end{smallmatrix} = 0 \tag{36}$$

with components

$$A = \circ \qquad F = Z\circ \qquad \psi_i = \boxdot \qquad \lambda_i = Z\boxdot \qquad A_{ij} = \boxdot\boxdot \qquad B_{ij} = Z\boxdot\boxdot \tag{37}$$

where A_{ij} and B_{ij} are symmetric in i, j. The content of (37) is that of the $Y = 0$ SU(2)-doublet, as obtained from (26) by addition of a further SU(2) index and reduction of the scalar fields into an SU(2) scalar and triplet. We may thus write

$$\Phi = \Phi_{0,2} + \Phi_{\frac{1}{2},1} \tag{38}$$

where $\Phi_{(n,m)}$ denotes an irrep with $Y = n$ and SU(2) dimension m. Similar analysis may be given for other superfields for $N = 2$ and for other N.

We may extend the super-tableaux calculus to the case of more than one central charge. This follows rather straightforwardly following the reduction of the symmetry along the lines of (14) and (14a). Thus for $N = 4$ with two central charges the basic superfield is in the (2, 2) representation which we can denote by $(\square, \square)$. The appropriate constraint to single out the $Y = 0$ fundamental irrep is

$$(\square\boxdot, \square) = (\square, \square\boxdot) = 0 \tag{39}$$

and the components will be constructed from the scalars $(\square, \square)$, spinors $\left(\begin{smallmatrix}\square\\ \boxdot\end{smallmatrix}, \square\right)$ and $\left(\square, \begin{smallmatrix}\square\\ \boxdot\end{smallmatrix}\right)$ and scalars and vectors $\left(\begin{smallmatrix}\square\\ \boxdot\end{smallmatrix}, \begin{smallmatrix}\square\\ \boxdot\end{smallmatrix}\right)$. The condition (19) does not prevent an infinite number of component multiplets from arising (Sohnius *et al* 1981), but these may be removed by a subsidiary condition to give quadrupling of the above components as an irreducible representation containing one arbitrary parameter. This irrep appears necessary to construct an off-shell version of $N = 4$ SGR (Bufton and Taylor 1983b). The three central charge situation can be handled similarly, with the fundamental irrep contained in the triplet SF $\square\square$ of SU(2) with constraint $\square\square\boxdot = 0$. The resulting components may again be read off by the super-tableaux,with smallest finite irrep being composed of eight duplicates of the components given directly by the super-tableaux.

The reduction of irreps to contain only a finite number of components can be seen in terms of the general SF solution of (3b) and (16) in the degenerate case $\kappa = 1$. This can be written as

$$\Phi(x^\mu, z_{ij}, z^*{}_{ij}, \theta_{\alpha+l}, \theta_{\alpha-}{}^l) = \int_C \mathrm{d}\lambda \, \exp[\lambda^{ij} z_{ij} + (\lambda^{-1})_{ij} z^*_{ij} p^2]$$
$$\times A(x, \lambda, \theta_{\alpha+l} - (\lambda^{-1})_{lm} (\not{p}\theta_-{}^m)_{\alpha+}) \tag{40}$$

where C is some contour in the $\frac{1}{2}N\ (N-1)$ complex dimensional space of antisymmetric matrices λ. For $N=2$, for example, this reduces to

$$\int_C \mathrm{d}\lambda\ \exp(\lambda z+\lambda^{-1}p^2z^*)A(x,\lambda,\psi_{\alpha+l}(\lambda)). \tag{41}$$

Since the action of $S_{\alpha+}{}^{l}$ of (20a) on (41) (or more generally (40)) is local in λ we may choose a single value of λ to obtain an irreducible representation. This corresponds in $N=2$ with $z=\frac{1}{2}(x_5+\mathrm{i}x_6)$ to $\partial_5^2=a\Box$, $\partial_6^2=(1-a)\Box$, and hence reduces the component content to a finite number.

5. Prepotentials

It is often stated that it is not possible to use superfields carrying central charges in the quantum context since they are constrained in such a way as to preclude the standard quantisation procedures. We have seen in § 4 that we can solve the Dirac condition (16) by means of the Fermi variable (23). It may be that this solution will prove difficult to non-linearise due to the non-locality explicitly present in the term involving $\not{p}^{-1}$, but that can only be determined by further more detailed investigation. We may also solve the wave equation (3b) in terms of superfields on four-dimensional space–time. Again we may have non-linearisation problems here, but again defer this to a more general analysis elsewhere. Our concern now is the discovery of prepotentials for the solution of the constraints of form (33), (34) or (36). We will only consider the $N=2$ cases (25) and (34) here since these are indicative of the form of solution in the other cases.

We will use the method of solution for $Z=0$ where the prepotential is expected to have the same spin and SU(2) transformations as the component field of highest (mass) dimension. The prepotential solution of (34) should therefore be an antisymmetric self-dual or anti-self-dual tensor $V_{\mu\nu}$. We may expect the solution to be of the form

$$\Phi=(\bar{D}_+{}^k\sigma_{\mu\nu}D_{+k})V^{\mu\nu}. \tag{42}$$

A straightforward use of (1b) allows us to show that (42) does indeed satisfy (34).

We now turn to the fundamental hypermultiplet defined by (25). As before we use the component of highest dimension, which is a spinor, so expect a solution to (25) of the form

$$\Phi^i=\bar{D}^{\alpha+i}\lambda_{\alpha+}. \tag{43}$$

The constraint (25) now becomes that of (34), so that if we use (42) our solution to (25) can be written as

$$\Phi^i=\bar{D}^{\alpha+i}(\bar{D}_+{}^k\sigma_{\mu\nu}D_{+k})(\sigma^{\mu\nu}\chi)_{\alpha+}. \tag{44}$$

We have thus been able to obtain prepotentials in both cases (25) and (34) along the lines expected in the non-central charge situation.

6. Discussion

We have shown that many of the features of non-central charge supersymmetry persist in the central charge situation. This may allow us to use central charge irreps more effectively to evade the no-go theorems (Rivelles and Taylor 1981, 1983a, Taylor 1982a). Before this can be done we must still resolve the question as to whether the remaining constraints, most specifically the Dirac constraint (16), do not prevent satisfactory use of the superfield framework for super-quantisation. For this we need to develop integration over superspace for our central charge irreps instead of using Lagrangians only at $\theta = 0$ (Sohnius 1978, Taylor 1980). This may now be possible, even in the $N = 2$ case, in terms of the unconstrained prepotentials of § 5.

Acknowledgments

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When the first two sections of this work had been completed we received a copy of a paper (Galperin *et al* 1982) with comparable results (though only for one central charge).

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